INTRODUCTION TO MULTIPLICITY THEORY

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1. Dimension theory and existence of multiplicity

1.1. Definitions.

Definition 1.1. We will use $(R, \mathfrak{m})$ to denote a *local* ring: R is commutative, Noetherian ring such that the set of all non-invertible elements forms an ideal, denoted $\mathfrak{m}$.

It follows that $R/\mathfrak{m}$ is a field, so $\mathfrak{m}$ is the unique maximal ideal of R. Examples: 1) the localization of any commutative, Noetherian ring at a prime ideal, e.g., the localization of $K[x_1, \ldots, x_d]$ at $(x_1, \ldots, x_d)$ where K is a field. 2) $R = K[[T_1, \ldots, T_d]]$ and quotients of it. 3) Geometrically: the ring of germs of functions at a point.

Definition 1.2. Let $(R, \mathfrak{m})$ be a local ring. An ideal I is $\mathfrak{m}$-primary if $\mathfrak{m}^n \subseteq I$ for some n.

Examples: $I = (x, y^2) \subset R = k[[x, y]]$. We have $\mathfrak{m}^2 \subset I$.

Definition 1.3. Recall that the length of an R-module is defined as

$$\ell(M) = \max\{L \mid \text{there is a chain } 0 = M_0 \subsetneq M_1 \subsetneq M_L = M\}.$$

It was proven in the Atiyah–MacDonald book ([1, Proposition 6.8]) that $\ell(M) < \infty$ if and only if M is Artinian and Noetherian. It follows that an ideal I of a local ring is $\mathfrak{m}$-primary if and only if R/I has finite length.

For example: if $I = (x, y^2) \subset R = k[[x, y]]$ then $\ell(R/I) = 2$ because $I \subset \mathfrak{m} \subset R$ is a saturated chain. Alternatively, $\ell(R/I) = \dim_k R/I = 2$ as we can use [1, Proposition 6.10] and the following remark.

Remark 1.4. In a module of finite length, if we take any maximal chain of submodules, then its length is the length of M[1]. By maximality in any such chain $M_{i+1}/M_i \cong R/\mathfrak{m}_i$, where $\mathfrak{m}_i$ is a maximal ideal. Thus $\ell(M) = \sum_{\mathfrak{m}} \ell_{R_\mathfrak{m}}(M_\mathfrak{m})$, where $\mathfrak{m}$ varies through all maximal ideals. Thus, in general little is lost by working in local rings.

We also recall the main property of the length.

Proposition 1.5 ([1, Proposition 6.9])**.** *Let R be a commutative ring. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of finite length R-modules. Then $\ell(M_2) = \ell(M_1) + \ell(M_3)$.*

Definition 1.6. Let $(R, \mathfrak{m})$ be a local ring and I be an $\mathfrak{m}$-primary ideal. The *Hilbert–Samuel* multiplicity of I is defined as

$$\mathrm{e}(I) = (\dim R)! \lim_{n \to \infty} \frac{\ell(M/I^n M)}{n^{\dim R}},$$

[1] [1, Proposition 6.7]

here $\dim R$ is the Krull dimension: the length of the longest chain of the prime ideals $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_L$.

We will now work to prove that this limit exists.

1.2. **Graded rings and Hilbert's polynomials.**

Definition 1.7. We will say that a commutative and unital ring G is *graded* if it has a decomposition $G = \oplus_{i \geq 0} G_i$ where G_i are abelian group (with respect to addition) such that $G_i \times G_j \subseteq G_{i+j}$. We will use notation $G_\bullet$ to specify the pieces.

Similarly, a graded module M over a graded ring $G_\bullet$ is a $G_\bullet$-module which can be decomposed as a direct sum $M = \sum_{i \in \mathbb{Z}} M_i$ of abelian groups such that $G_i M_j \subseteq M_{i+j}$.

It follows from the definition that G_0 is a ring and M_i are G_0-modules. Recall that a Noetherian and Artinian module has finite length. Thus, if G_0 is an Artinian ring and G_i are finitely generated G_0-modules, then G_n have *finite length* as G_0-modules. Our goal is to study how $\ell(G_n)$ depends on n.

Example 1.8. Let R be a ring. A polynomial ring $G = R[T_1, \dots, T_d]$ is graded by the usual degree:

$$G_i := \{\text{homogeneous polynomials of degree } i\}.$$

We can also make graded G-modules by twisting the grading: $G(\ell)_i = G_{\ell+i}$ defines a new graded module.

This motivates the following notation.

Definition 1.9. An element x of a graded ring is homogeneous if it belongs to one of the graded pieces G_i. An ideal I of a graded ring is homogeneous if it can be generated by homogeneous elements (not necessarily of same degree).

Example 1.10. In a polynomial ring $k[x, y]$ with the usual grading, ideals $(x, y^2), (x^2 + y^2)$ are homogeneous, while $(x + 1), (x^2 + y)$ are not.

Exercise 1.11. If $m \in M_i$ is a homogeneous element, then $\operatorname{Ann} m := \{x \in G_\bullet \mid xm = 0\}$ is a homogeneous ideal.

Definition 1.12. A (graded) *homomorphism* of graded $G_\bullet$-modules $M_\bullet$ and $N_\bullet$ is a homomorphism $f\colon M_\bullet \to N_\bullet$ as non-graded modules that preserves the grading: $f(M_i) \subseteq N_i$.

We will need the existence of graded prime filtrations.

Lemma 1.13. *Let M be a finitely generated graded module over a Noetherian graded ring $G_\bullet$. There exists a filtration $0 = N_0 \subset N_1 \subset \cdots \subset N_r = M$ of graded submodules such that $N_i/N_{i-1} \cong (S/\mathfrak{p}_i)(\ell_i)$, where $\mathfrak{p}_i$ is a homogeneous prime ideal and $\ell_i \in \mathbb{Z}$.*

Proof. First, we claim that M has a homogeneous associated prime. Consider a maximal element $\mathfrak{p}$ of the set of ideals of the form $\operatorname{Ann} x$ where $0 \neq x \in M$ is homogeneous. We claim that $\mathfrak{p}$ is prime.

Suppose that $ab \in \mathfrak{p} = \operatorname{Ann} m$ and write their decomposition in the homogeneous components $a = \sum a_i, b = \sum b_i$. We can write $ab = \sum f_k x_k$, where x_i are a system of homogeneous generators of M. If a_{i_0}, b_{j_0} are smallest degree nonzero components of a and b, we can separate the terms of degree $i_0 + j_0$ in ab to see that $a_{i_0} b_{j_0} \in \mathfrak{p}$. Thus it suffices to assume that a, b are homogeneous: if we prove that $a_{i_0}, b_{j_0} \in \mathfrak{p}$, we remove them and continue.

Now, if $b \notin \mathfrak{p}$, then $\mathfrak{p} \subseteq \operatorname{Ann} bm$, so they must be equal by maximality. Because, $a \cdot bm = 0$, $a \in \mathfrak{p}$. This finishes the claim.

Second, define a map $(S/\mathfrak{p})(-\deg m)$ to M by sending $1 \to m$. This map is an inclusion because $\mathfrak{p} = \operatorname{Ann} m$ and it is an inclusion of *graded modules* due to the shift. Thus, we may let $N_1 = (S/\mathfrak{p})(-\deg m)$- And proceed to build N_2 by induction, by taking an associated prime in M/N_i we find N_{i+1}. This process terminates by the Noetherian assumption. ☐

Remark 1.14. This is essentially the same proof that is used to prove existence of prime filtrations for non-graded modules. This result is recovered by giving R and M zero grading.

Theorem 1.15 (Hilbert). *Let $G_\bullet$ be a graded ring such that*

(1) *$A = G_0$ is an Artinian ring,*
(2) *G_1 is a finitely generated G_0-module,*
(3) *G is generated by G_1 as an algebra over G_0.*

Then for any finitely generated graded $G_\bullet$-module $M_\bullet$ there exists a polynomial $P(T) \in \mathbb{Q}[T]$ and an integer N such that $\ell_A(M_n) = P(n)$ for all $n \geq N$.

Proof. By definition of graded homomorphism, an exact sequence of graded modules

$$0 \to L \to M \to N \to 0$$

gives the equality $\ell_A(M_n) = \ell_A(N_n) + \ell_A(L_n)$. Thus Lemma 1.13 shows that it suffices to prove the theorem for $M = (G/\mathfrak{p})(\ell)$, since $\ell_A(M(\ell)_n) = \ell_A(M_{n+\ell})$ it suffices to consider $M = G/\mathfrak{p}$.

If $\mathfrak{p}$ contains all of $\oplus_{i \geq 1} G_i$, then $M = G/\mathfrak{p}$ has nothing in positive degree, so $\ell_A(M_n) = 0$ for $n \geq 1$ in this case. Note that in this case $\dim M = 0$, we set 0 to have degree -1 by convention.

Otherwise, there is a homogeneous element $x \notin \mathfrak{p}$ of positive degree 1. We then have the exact sequence

$$0 \to (G/\mathfrak{p})(-1) \xrightarrow{1 \mapsto x} G/\mathfrak{p} \to G/(\mathfrak{p}, x) \to 0$$

which gives that

$$\ell_A((G/(\mathfrak{p}, x))_n) = \ell_A((G/(\mathfrak{p}))_n) - \ell_A((G/(\mathfrak{p})_{n-1}).$$

It is not hard to check, that if $\ell_A((G/(\mathfrak{p}, x))_n)$ is given by a polynomial $f(T)$ of degree d then

$$\ell_A((G/(\mathfrak{p}))_n) = \ell_A((G/(\mathfrak{p}))_{n_0}) + \sum_{k=n_0}^{n} f(n)$$

is a polynomial[2] of degree $d+1$ for large n.

This allows us to finish the proof by induction on $\ell_A(G_1)$. Namely, since $x \in G_1$, $\ell_A(G/(\mathfrak{p}, x))_1) < \ell_A(G_1)$. ☐

[2]One can use the binomial coefficients to see this, see below.

1.2.1. *Coefficients of the Hilbert polynomial.* If G is not artinian, $\ell_A(G_i) > 0$, so the Hilbert polynomial has positive leading coefficient. But, this polynomial also take integer values so it has a special representation. To do so, I want to recall a few properties of binomial coefficients:

(1) $n \mapsto \binom{n+d}{d}$ is a polynomial in n of degree d. By expansion its leading coefficients are $\frac{1}{d!}n^d + \frac{d+1}{2(d-1)!}n^{d-1} + \cdots$.
(2) Polynomials $\binom{n+k}{k}$, $0 \leq k \leq d$ form a basis of the vector space of polynomials of degree at most d.
(3) Polynomials $\binom{n+d}{d}$, $d \geq 1$, take only positive integer values if $n \in \mathbb{Z}_{>0}$.
(4) We have a recurrence identity $\binom{n}{d} - \binom{n-1+d}{d} = \binom{n+d-1}{d-1}$.

Proposition 1.16. *In the setting of Theorem 1.15 we may write the Hilbert polynomial $P(T)$ as $n \mapsto \ell_A(G_i)$ as*

$$P(T) = \sum_{k=0}^{d} e_k \binom{T+d-k}{d-k}$$

where d is the degree of $P(T)$ and e_k are integers, called Hilbert coefficients*. Moreover, if $d \geq 0$, then the leading coefficient e_0 is positive.*

Proof. This can be proven as a part of Hilbert's theorem by following its proof by using the recurrence for binomial coefficients. We know that the leading coefficient is positive because the values at all large integers are positive.

Alternatively, this is true more generally. If the value of $P(n)$ are integers for $n \in \mathbb{Z}_{n \geq n_0}$, then we obtain that $P(t) \in \mathbb{Q}[t]$ (say by interpolation formulas). The polynomials $\binom{t+i}{i}$ form a basis of $\mathbb{Q}[t]$, so there is a decomposition with $e_k \in \mathbb{Q}$. Then one can use the condition on the values and the induction that e_k. $\square$

1.3. **Constructions of graded rings.** The proof of the Hilbert theorem gives a bound $\ell(G_1)$ on the degree of the polynomial, but we aim to show that it is equal to $\dim G - 1$. But first we connect Hilbert's theorem to multiplicity. To do so we introduce two construction of graded rings from a local ring.

1.3.1. *Rees algebra and Artin–Rees lemma.* In the following we will need to use several times the Artin–Rees lemma, so let me give you a proof. This is largely same as [1, Proposition 10.9]

Proposition 1.17 (Artin–Rees)**.** *Let A be a Noetherian ring, I be an ideal, M be a finitely generated A-module, and M' be its submodule. Then there exists an integer c such that $I^nM \cap M' = I^{n-c}(I^cM \cap M')$*

Proof. First, consider the ring $R[IT] = \oplus^n I^n T^n \subseteq R[T]$, where T is a formal variable[3]. This is a graded ring. Since R is Noetherian, there are finitely many elements $a_1, \ldots, a_m$ that generate I then $a_1T, \ldots, a_mT$ generate $R(I)$ as an algebra over R. So $R[IT]$ is also Noetherian.

[3]The variable T helps us to distinguish the element, this way we can distinguish $I \subseteq R$ from IT, the degree 1 piece.

We introduce modules over this ring in a natural way:

$$N' = \oplus_n (M' \cap I^n M)T^n \subseteq \oplus_n I^n M T^n = M \oplus ITM \oplus I^2T^2M \dots .$$

Observe that if m_i generate M then they also generate $\oplus_n I^n T^n M$ over the ring $R(I)$, so it follows that N' is finitely generated. Now, let $x_1, \dots, x_m$ be generators of this module. By breaking them into pieces we may assume they are homogeneous. Let now c be the maximum of the degrees of x_i. Then $(M' \cap I^n M)T^n$ is the degree n piece of N' and we must be able to write any element y of it using $x_1, \dots, x_m$:

$$y \in (IT)^{n-\deg x_1} x_1 + \dots + (IT)^{n-\deg x_m} x_m \in I^{n-c}T^{n-c}N_c.$$

Thus $M' \cap I^n M \subseteq I^{n-c}(I^c M \cap M')$, the opposite inclusion is clear. $\square$

Definition 1.18. The ring $R[IT]$ is called the *Rees algebra* of I.

1.3.2. *Associated graded rings.*

Definition 1.19. Let R be a ring and I be an ideal. The associated graded ring of I is defined as

$$\operatorname{gr}_I(R) := \bigoplus_{n \geq 0} I^n / I^{n+1}, \text{ where } I^0 := R.$$

Similarly, the associated graded module is

$$\operatorname{gr}_I(M) := \bigoplus_{n \geq 0} I^n M / I^{n+1} M, \text{ where } I^0 M := M.$$

The multiplication on $\operatorname{gr}_I(R)$ is inherited from R: if $\bar{a} \in I^n/I^{n+1}, \bar{b} \in I^m/I^{m+1}$ with lifts $a \in I^n, b \in I^m$ then $ab \in I^{n+m}$ and we define $\bar{a} \cdot \bar{b} = ab + I^{n+m1}$. It is easy to check that this does not depend on the lifts of $\bar{a}, \bar{b}$. This ring structure also gives us that $\operatorname{gr}_I(R) \cong R[IT]/IR[IT]$.

Exercise 1.20. Confirm that $\operatorname{gr}_I(R)$ is a graded ring which is generated as an algebra in degree 1 over R/I, its degree 0 part. In M is finitely generated, check that $\operatorname{gr}_I(M)$ is a finitely generated $\operatorname{gr}_I(R)$-module.

Via this exercise we may apply Theorem 1.15 to $\operatorname{gr}_I(M)$.

Corollary 1.21. *Let $(R, \mathfrak{m})$ be a local ring, M be a finitely generated R-module, and I be an ideal such that $I + \operatorname{Ann} M$ is $\mathfrak{m}$-primary. Then there is a polynomial $P_I(T) \in \mathbb{Q}[T]$ and an integer n_0 such that $\ell(M/I^n M) = P_I(n)$ for $n \geq n_0$. The degree of this polynomial, called* the Hilbert–Samuel polynomial *of I, is independent of I.*

Moreover, if we let d to denote this common degree, then we can decompose

$$P_I(n) = \sum_{k=0}^{d} e_k(I) \binom{n+d-k}{d-k},$$

where $e_k(I) \in \mathbb{Z}$ and $e_0 \geq 1$. In addition, for all large n,

$$\ell(I^n M / I^{n+1} M) = \sum_{k=1}^{d-d} e_k(I) \binom{n+d-k-1}{d-k-1}.$$

Proof. We can replace R by $R/\operatorname{Ann} M$, now I is $\mathfrak{m}$-primary and we may apply Theorem 1.15 in $\operatorname{gr}_I(M)$. Thus there is a polynomial $Q(T)$ and an integer n_0 such that $\ell(I^nM/I^{n+1}M) = Q(n)$ for $n \geq n_0$. By Proposition 1.16 it has the required binomial decomposition. But then for $n \geq n_0$

$$P_I(n) = \ell(M/I^nM) = \ell(M/I^{n_0}M) + \sum_{K=n_0}^{n-1} Q(k)$$

is a polynomial of degree $\deg Q + 1$ and has the required form due to the binomial identities.

Second, we prove that the degree of $P_I(n)$ does not depend on I. Observe that there exists an integer c such that $\mathfrak{m}^c \subseteq I \subseteq \mathfrak{m}$. Therefore, we have inequalities

$$\ell(M/\mathfrak{m}^{cn}) \geq \ell(M/I^nM) \geq \ell(M/\mathfrak{m}^nM).$$

Now, if $\ell(M/\mathfrak{m}^nM) = P_\mathfrak{m}(n)$ is a polynomial of degree d then $\ell(M/\mathfrak{m}^{cn}M) = P_\mathfrak{m}(cn)$ is also a polynomial of degree d so it follows that $P_I(n)$ is also a polynomial of degree d. $\square$

Remark 1.22. Let $(R, \mathfrak{m})$ be a local ring and I be an $\mathfrak{m}$-primary ideal. For an element $x \in R$ we will use x^* to denote its image in $\operatorname{gr}_I(R)$. If $I \neq 0$ then there exists an integer m such that $x \in I^m \setminus I^{m+1}$ ($I^m \neq I^{m+1}$ and $\cap I^n = 0$ [1, Corollary 10.19] by Nakayama's lemma). Then $x^* = x + I^{m+1}$ as an element of $I^m/I^{m+1} \subset \operatorname{gr}_I(R)$.

Exercise 1.23. Now let J be an arbitrary ideal. Show that $J^* = \{x^* \mid x \in J\}$ is equal to $\oplus_{k\geq 0} \frac{J \cap I^k + I^{k+1}}{I^{k+1}}$. Show that $\operatorname{gr}_I(R/J) \cong \operatorname{gr}_I(R)/J^*$.

1.4. **Applications in the dimension theory.** We now want to link the dimension of the Hilbert polynomial and the Krull dimension. Let us denote $d_{Hilb}(M)$ to be the degree of Hilbert–Samuel polynomials $\ell(M/I^nM)$, it does not depend on I by Corollary 1.21.

We present the following properties of this invariant.

Proposition 1.24. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M, N be finitely generated R-modules.*

(1) *If $N \to M$ is injective, then $d_{Hilb}(N) \leq d_{Hilb}(M)$.*
(2) *If $M \to N$ is surjective, then $d_{Hilb}(N) \leq d_{Hilb}(M)$.*
(3) *If $x \in \mathfrak{m}$ is not a zerodivisor on M, then $d_{Hilb}(M/xM) \leq d_{Hilb}(M) - 1$.*

Proof. Let I be an arbitrary $\mathfrak{m}$-primary ideal. First, if $M \to N$ is surjective, then $M/I^nM \to N/I^nN$ is surjective too, so the inequality on degrees follow. $d_{Hilb}(M') \leq d_{Hilb}(M)$.

Second, we have an exact sequence

$$0 \to N/(I^nM \cap N) \to M/I^nM \to M/(I^nM + N) \to 0$$

which shows that $\ell(N/(\mathfrak{m}^nM \cap N)) \leq \ell(M/\mathfrak{m}^nM)$. By the Artin–Rees lemma, there is a constant c such that $\mathfrak{m}^nM \cap N \subseteq \mathfrak{m}^{n-c}N$, so $\ell(N/I^{n-c}N) \leq \ell(M/I^nM)$ and the inequality on degrees follows.

For the last assertion plug $N = xM$ in the exact sequence above. Then by the Artin–Rees lemma there is $c > 0$ for which we have inequalities

$$\ell(M/\mathfrak{m}^nM) - \ell(M/(\mathfrak{m}^nM + N)) \geq \ell(N/\mathfrak{m}^{n-c}N).$$

Since the map $M \to N$ given by the multiplication by x is injective as x is not a zerodivisor, it is an isomorphism. Therefore, if $\ell(M/\mathfrak{m}^n M)$ is given by the polynomial $P(n)$ then

$$P(n) - \ell(M/(\mathfrak{m}^n M + N)) \geq P(n-c)$$

and it follows that

$$\lim_{n\to\infty} \frac{\ell(M/(\mathfrak{m}^n M + N))}{n^{d_{Hilb}(M)}} = 0$$

so it is a polynomial of smaller degree. □

Lemma 1.25. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and suppose that $I = (x_1, \ldots, x_d)$ is an $\mathfrak{m}$-primary ideal. Then $d_{Hilb}(R) \leq d$.*

Proof. Observe that the R/I-module I^n/I^{n+1} can be generated by monomials in $x_1, \ldots, x_d$ of degree n. Therefore, there is a surjection

$$\oplus^{\binom{n+d-1}{d-1}} S/I \to I^n/I^{n+1} \to 0$$

and $\ell(I^n/I^{n+1}) \leq \binom{n+d-1}{d-1}\ell(S/I)$. Thus the Hilbert polynomial of S has degree at most $d-1$ and the assertion follows. □

Theorem 1.26 (The main theorem of dimension theory)**.** *Let $(R, \mathfrak{m})$ be a local ring. The following 3 numbers coincide:*

1. *The Krull dimension of R, $\dim R$.*
2. *The minimal number of elements of $\mathfrak{m}$ that are needed to generate some $\mathfrak{m}$-primary ideal, $\delta(R)$.*
3. *The degree of the Hilbert–Samuel polynomial of any $\mathfrak{m}$-primary ideal, $d_{Hilb}(R)$.*

Proof. Let us prove that $d_{Hilb}(R) \geq \dim(R)$. We use induction on $d(R)$. In the base case, $\ell(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = 0$ for large n, so R is Artinian by the Nakayama lemma. Suppose that $\dim R = L > 0$ and consider a maximal chain of prime ideals

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_L = \mathfrak{m}. \tag{1}$$

Choose elements $x_i \in \mathfrak{p}_i \backslash \mathfrak{p}_{i-1}$. Then by Proposition 1.24 $d_{Hilb}(R/(\mathfrak{p}_i, x_{i+1})) \leq d_{Hilb}(R/\mathfrak{p}_i) - 1$. Since clearly $d_{Hilb}(R/p_{i+1}) \leq d_{Hilb}(R/(\mathfrak{p}_i, x_{i+1}))$ by surjection, we obtain that $0 \leq d_{Hilb}(R) - L = d_{Hilb}(R) - \dim R$.

Via Lemma 1.25 $d_{Hilb}(R) \leq \delta(R)$, so it remains to show that $\delta(R) \leq \dim(R)$. We use induction on $\dim R$. The base case $\dim R = 0$ is vacuous: 0 is already $\mathfrak{m}$-primary. Now suppose $\dim R > 0$. Since there are finitely many minimal prime ideals $\mathfrak{p}_i$, by prime avoidance we can find $x_1 \in \mathfrak{m} \setminus \cup_i \mathfrak{p}_i$. Because any saturated chain of prime ideals must start with a minimal prime, $\dim R/x_1 R \leq \dim R - 1$. By induction we can find at most $\dim R - 1$ elements $x_2, \ldots, x_d$ that generate an $\mathfrak{m}R/x_1 R$-primary ideal. Then the ideal $x_1, \ldots, x_d$ is $\mathfrak{m}$-primary. This proves that $d \leq \dim R$. □

Corollary 1.27. *Let $(R, \mathfrak{m})$ be a local ring and I be an $\mathfrak{m}$-primary ideal. Let $G = \mathrm{gr}_I(R)$ and $M = \mathfrak{m}/I \oplus I/I^2 \oplus \cdots$. Then M is a maximal ideal of G and the height of M is $\dim R$.*

Proof. Clearly, $G/M \cong R/\mathfrak{m}$ so it is a maximal ideal. In order to compute its height, we take an ideal $G_+ = \oplus_{i>0} G_i$ in G. Since $G/G_+ \cong R/I$ the ideal G_+ is $M = \mathfrak{m}/I \oplus I/I^2 \oplus \cdots$-primary.

We have isomorphism $G_+^n/G_+^{n+1} \cong I^n/I^{n+1}$ which shows that the Hilbert polynomial of $G_+ = \oplus_{i>0} G_i$ in G is the same as the Hilbert polynomial of I in R. $\square$

Corollary 1.28. *Let k be a field. Then* $\dim k[[x_1, \dots, x_d]] = d$.

Proof. Note that $R = k[[x_1, \dots, x_d]]$ is a local ring and $\operatorname{gr}_{\mathfrak{m}}(R) = k[x_1, \dots, x_d]$ graded by the usual degree. Then R_n is a k-vector space with a basis of all monomials of degree n. The number of such monomials is $\binom{n+d-1}{d-1}$ (splitting n objects in d piles). $\square$

Corollary 1.29 (Krull's height theorem)**.** *Let R be a Noetherian ring and $x_1 \dots, x_r \in R$ to generate a proper ideal. Then any minimal prime $\mathfrak{p}$ of $(x_1, \dots, x_r)$ has height at most r.*

Proof. We localize at $\mathfrak{p}$ and reduce to the theorem. $\square$

We now want to prove the corresponding statement for modules. This essentially reduces to the case of rings, but requires a discussion.

Definition 1.30. If M is an R-module we define $\operatorname{Supp} M = \{\mathfrak{p} \in \operatorname{Spec} R \mid M_{\mathfrak{p}} \neq 0\}$ and $\operatorname{Ann} M = \{x \in R \mid xM = 0\}$, it is easy to see that $\operatorname{Ann} M$ is an ideal.

Remark 1.31. [4] Clearly, $x \in \operatorname{Ann} M \setminus \mathfrak{p}$ then $xM_{\mathfrak{p}} = 0$, so $\operatorname{Supp} M \subseteq V(\operatorname{Ann} M) = \operatorname{Spec} R/\operatorname{Ann} M$. If M is finitely generated we can easily prove that $\operatorname{Supp} M = \operatorname{Spec} R/\operatorname{Ann} M$. Namely, let $x_1, \dots, x_n$ generate M. If $M_{\mathfrak{p}} = 0$, then by definition there are elements $s_i \notin \mathfrak{p}$ such that $s_i x_i = 0$. But then $s_1 \cdots s_n \in \operatorname{Ann} M \setminus \mathfrak{p}$.

In fact, if M is finitely generated, $\operatorname{Supp} M/xM = \operatorname{Supp} M \cap V(x) = V(\operatorname{Ann} M + (x))$ for any element x. We know that $V(\operatorname{Ann} M + (x)) = V(x) \cap V(\operatorname{Ann} M)$ by properties of $V(I)$. We also have that $\operatorname{Ann} M + (x) \subseteq \operatorname{Ann} M/xM$ which gives one containment. For the converse, let $\mathfrak{p}$ be a prime containing $x + \operatorname{Ann} M$. Then $M_{\mathfrak{p}} \neq 0$ and $x \in \mathfrak{p}R_{\mathfrak{p}}$, so by Nakayama's lemma $M_{\mathfrak{p}}/xM_{\mathfrak{p}} = (M/xM)_{\mathfrak{p}}$ is not zero. The assertion is now clear by comparing chains in $\operatorname{Supp} M$ and $\operatorname{Supp} M \cap V(x)$.

Corollary 1.32. *Let $(R, \mathfrak{m})$ be a local ring and M be a finitely generated R-module. The following* 3 *numbers coincide:*

(1) *The Krull dimension* $\dim M$ *of M, i.e., the Krull dimension of* $\operatorname{Supp} M$.
(2) *The degree of the Hilbert–Samuel polynomial $n \mapsto M/I^nM$ for any ideal I such that* $\operatorname{Ann} M + I$ *is $\mathfrak{m}$-primary.*
(3) *The minimal number $\delta(M)$ such that there are elements $x_1, \dots, x_d \in \mathfrak{m}$ for which the quotient module $M/(x_1, \dots, x_d)M$ is Artinian.*

Proof. Observe that replacing R by $R' = R/\operatorname{Ann} M$ does not change any of these numbers due to Remark 1.31 and the fact that $\ell_R(M/I^nM) = \ell_{R'}(M/I^nM)$ as these quotient are R'-modules. Now, by the same remark, $\dim M = \dim R'$ and $\delta(M) = \delta(R')$ due to the topological characterization. Hence, $\dim(M) = \delta(M)$.

Since M is finitely generated, there is a free module $\oplus^n R'$ that surjects onto M. It follows that $\ell(M/\mathfrak{m}^nM) \leq n\ell(R'/\mathfrak{m}^nR')$, so $d_{Hilb}(M) \leq d_{Hilb}(R') = \dim R' = \dim M$. For the opposite inequality[5], let $\mathfrak{p}$ be a minimal prime of R' such that $\dim R'/\mathfrak{p} = \dim R'$. Since

[4] This is [1, Exercise 3.19]

[5] Alternatively, we can localize a prime filtration of M to observe that $R'/\mathfrak{p}$ must appear as one of the quotients. But then $d_{Hilb}(R'/\mathfrak{p}) \leq d_{Hilb}(M)$ by Proposition 1.24.

$\operatorname{Supp} M = \operatorname{Supp} R'$, this is also a minimal prime of M, hence an associated prime. Thus we have an injection $N = R'/\mathfrak{p} \to M$. By Proposition 1.24, $d_{Hilb}(M) \geq d_{Hilb}(R'/\mathfrak{p}) = \dim R'/\mathfrak{p} = \dim R' = \dim M$. ☐

We record an important corollary.

Corollary 1.33. *If $(R, \mathfrak{m})$ is a Noetherian local ring and M be a finitely generated R-module. If $x \in \mathfrak{m}$ then $\dim M - 1 \leq \dim M/xM \leq \dim M$.*

Moreover, $\dim M/xM = \dim M$ if and only if x is contained in some minimal prime ideal $\operatorname{Ann} M \subseteq \mathfrak{p}$ such that $\dim R/\mathfrak{p} = \dim M$.

Proof. First, we know that M/xM is an $R/\operatorname{Ann} M$-module so $\dim M/xM \leq \dim R/\operatorname{Ann} M = \dim M$. For the other inequality, suppose that $\dim M/xM < \dim M - 1$, then by applying Corollary 1.32 in $\dim M/xM$ we get a contradiction with Corollary 1.32 in M as we will need too few elements to get $\dim M/(x_1, \ldots, x_d)M = 0$ (Note that a finitely generated Artinian module is annihilated by a power of the maximal ideal so its dimension is 0).

The second assertion follows from the formula $\operatorname{Supp} M/xM = V(\operatorname{Ann} M + (x))$. ☐

Definition 1.34. Let $(R, \mathfrak{m})$ be a local ring and M be a finitely generated R-module of dimension $d \geq 1$. We say that elements $x_1, \ldots, x_d$ form a system of parameters on M if $\dim M/(x_1, \ldots, x_d)M = 0$.

We say that an element $x \in R$ is a parameter on M if $\dim M/xM = \dim M - 1$.

We have proven that systems of parameters and parameter elements always exist.

1.5. **Existence of multiplicity and first examples.** Now, by combining Theorem 1.26, Corollary 1.21, and Proposition 1.16 we derive the existence of multiplicity.

Theorem 1.35. *Let $(R, \mathfrak{m})$ be a local ring, M be a finitely generated R-module, and I be an ideal such that $\operatorname{Ann} M + I$ is $\mathfrak{m}$-primary. Then*

$$\lim_{n\to\infty} (\dim M)! \frac{\ell(M/I^n M)}{n^{\dim M}} = \lim_{n\to\infty} (\dim M - 1)! \frac{\ell(I^n M/I^{n+1} M)}{n^{\dim M - 1}} \in \mathbb{Z}_{>0}.$$

This limit is the leading coefficient of the Hilbert–Samuel polynomial multiplied by $(\dim M)!$.

Remark 1.36. This limit might not be the Hilbert–Samuel multiplicity of M! Recall that we define it as

$$\mathrm{e}(I; M) = \lim_{n\to\infty} (\dim R)! \frac{\ell(M/I^n M)}{n^{\dim R}}.$$

Thus, the power in the denominator makes the limit to be 0 when $\dim R > \dim M$, this will make multiplicity additive in short exact sequences. In order to work with the leading coefficient of the Hilbert–Samuel polynomial of M, we can pass to $R' = R/\operatorname{Ann} M$ and take the multiplicity of M as an R'-module.

Exercise: Let $(R, \mathfrak{m})$ be a local ring, show that $\mathrm{e}(\mathfrak{m}) = \mathrm{e}((\mathfrak{m}, T))$ where the latter is the maximal ideal of $R[[T]]$.

Since we defined the Hilbert function through the associated graded ring, the multiplicity can be compute there. The following result is an easy consequence of the proof of Corollary 1.27.

Corollary 1.37. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring and* I *be an* $\mathfrak{m}$*-primary ideal. Let* $G = \mathrm{gr}_I(R)$ *and* $G_+ = \oplus_{k\geq 1} I^k/I^{k+1}$ *an ideal of* G*. Then* $\mathrm{e}(I; R) = \mathrm{e}(G_+; G)$*.*

Example 1.38. Let $R = k[[x_1, \ldots, x_d]]$ and $\mathfrak{m} = (x_1, \ldots, x_d)$. Then $R/\mathfrak{m}^{n+1}$ has a k-basis of monomials of degree at most n. Therefore, $\ell(R/\mathfrak{m}^{n+1}) = \binom{n+d}{d}$ and the multiplicity is 1.

Example 1.39. Let $R = k[[x_1, \ldots, x_d]]$ and f be a homogeneous element. Note that $R_n = \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is spanned by all monomials of degree n. Observe that

$$0 \to R(-\deg f) \xrightarrow{\times f} R \to R/fR \to 0$$

is a map of graded R-modules as f is homogeneous. Thus we have an exact sequence on homogeneous components for $n > \deg f$

$$0 \to R_n \to R_{n+\deg f} \to (R/fR)_{n+\deg f} \to 0.$$

Thus we can compute the Hilbert function of R/fR as $\binom{n+\deg f+d-1}{d-1} - \binom{n+d-1}{d-1}$. Note that we may expand

$$\binom{n+m}{d} = \frac{(n+m)\cdots(n+m-d+1)}{d!} = \frac{n^d}{d!} + \frac{d(2m-d+1)}{2d!}n^{d-1} + O(n^{d-2})$$

This implies that $\dim_k(R/fR)_{n+\deg f} = \deg f n^{d-2}/(d-2)! + O(n^{d-3})$, so $\mathrm{e}(R/fR) = \deg f$.

1.5.1. *Regular rings.*

Definition 1.40. A Noetherian local ring $(R, \mathfrak{m})$ is regular if there exist elements $x_1, \ldots, x_d$, $d = \dim R$ such that $\mathfrak{m} = (x_1, \ldots, x_d)$.

Note that the maximal ideal cannot be generated by less than d elements by Krull's height theorem, so this is an extremal condition.

Examples: a field, a DVR ([1, Chapter 9]) are regular local rings. This property is preserved by adding variables, so $K[[T_1, \ldots, T_d]]$, where K is a field (or a DVR), is a regular local ring.

We proved that the multiplicity of a power series ring is 1 and this extends to regular rings.

Corollary 1.41. *Let* $(R, \mathfrak{m})$ *be regular local ring. Then* $\mathrm{e}(\mathfrak{m}) = 1$ *and* R *is a domain.*

Proof. Let $k = R/\mathfrak{m}$ and $\mathfrak{m} = (x_1, \ldots, x_d)$. We claim that $\mathrm{gr}_{\mathfrak{m}}(R) \cong k[T_1, \ldots, T_d]$ (a polynomial ring!) by sending $T_i \mapsto x_i + \mathfrak{m}^2$. We know that this map is surjective since x_i generate $\mathrm{gr}_{\mathfrak{m}}(R)$, and there cannot be a nonzero polynomial in the kernel since the dimensions match. Thus the computation reduces to combinatorics as in Example 1.38. Last, we observe that R is a domain because $\mathrm{gr}_{\mathfrak{m}}(R)$ is a domain (exercise![6]). □

Exercise: extend Example 1.39 to an arbitrary regular local ring $(R, \mathfrak{m})$ by showing that $\mathrm{e}(\mathfrak{m}, R/fR) = \mathrm{ord}\, f$, where $\mathrm{ord}(f) := \min\{n \mid f \in \mathfrak{m}^n\}$. Hint: use that $\mathrm{gr}_{\mathfrak{m}}(R/fR) = \mathrm{gr}_{\mathfrak{m}}(R)/f^* \mathrm{gr}_{\mathfrak{m}}(R)$, where f^* is the image of f in $\mathrm{gr}_{\mathfrak{m}}(R)$.

[6] [1, Lemma 11.23]

References

[1] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

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